Updating Awareness and Information Aggregation*

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Abstract

The ability of markets to aggregate information through prices is examined in a dynamic environment with unawareness. We find that if all traders are able to minimally update their awareness when they observe a price that is counterfactual to their private information, they will eventually reach an agreement, thus generalising the result of Geanakoplos and Polemarchakis [1982]. Moreover, if the traded security is separable, then agreement is on the correct price and there is information aggregation, thus generalizing the result of Ostrovsky [2012] for non-strategic traders. We find that a trader increases her awareness if and only if she is able to become aware of something that other traders are already aware of and, under a mild condition, never becomes aware of anything more. In other words, agreement is more the result of understanding each other, rather than being unboundedly sophisticated.

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1 Introduction

Do markets aggregate information through prices? This question has been examined at least since Hayek [1945], as information aggregation is considered one of the most desirable properties that a market can have. Intuitively, the mechanism is simple. A highly priced security should prompt traders to sell, if they believe that its expected value is low. But their sell orders reveal to the market part of their private information,

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thus prompting everyone else to update and either sell or buy, revealing further information. As long as there is enough time, price movements should eventually aggregate all available information, so that the price converges to the true value of the security.

Information aggregation has been studied in various settings, for example using the Rational Expectations Equilibria in large decentralised markets. Moreover, Ostrovsky [2012] showed that even if there are few and large strategic traders, markets aggregate information, as long as securities are "separable".

In recent years, the creation of numerous *prediction markets* aims exactly at leveraging this property, in order to provide better predictions about future events, using the wisdom of the crowd. Examples of firms that have used internal prediction markets are Google, Microsoft, Ford, General Electric and HP (O'Leary [2011]). Iowa Electronic Markets (IEM) is run by the University of Iowa and aims at predicting political events, in many cases with considerable success. For example, Berg et al. [2008] estimates that for the five presidential elections between 1988 and 2004, 74% of the time the IEM outperformed the predictions of 964 polls, whereas for predictions 100 days in advance, the IEM outperformed at every election.

The purpose of this paper is to examine whether information aggregation and agreement is still possible in an environment with unawareness, where the information structure is not common knowledge. The intuitive mechanism that was explained above implicitly assumed that if trader i observes trader j selling the security, she can correctly identify what this reveals about her private information. However, if i is unaware of something that j is aware of, this might not be possible. For example, suppose that trader i is unaware of the concept of interest rates, hence she cannot understand that j has some information about whether they will go up or down. As a result, she cannot interpret j's sale order as a signal that the interest rates are about to increase.[1]

If trader i cannot rationalise j's actions, there are two possibilities. The first is that she ignores the information revealed by j's action, thinking she might be wrong or irrational. Effectively, she starts behaving like a "noise trader", ignoring aspects of her environment. In such a case, we cannot expect to have agreement or information aggregation.

The other possibility is that she tries hard to rationalise j's action and, in the process, she manages to increase her awareness. We assume that awareness updating is minimal, so that she never becomes more aware than necessary in order to rationalise j's action. Our main result, Theorem 1, specifies that if traders are always able to minimally update their awareness, then they will eventually agree on the price of the security. This result is effectively a generalization of Geanakoplos and Polemarchakis [1982], who showed that if two traders with common prior take turns in announcing their posterior about an event, eventually they will agree. Moreover, if the security is separable, then they will also agree on the correct price so that there is information aggregation. Ostrovsky [2012] showed the same result in an environment without unawareness, for the case of non-strategic traders. A security is separable given an information structure if, for every non-degenerate prior on the values of the security,

[1] The connection between low awareness and the ability to correctly reason about the information of others was first explored in Galanis [2011a, 2013].

there is a trader who receives an informative signal with positive probability. The most well-known example is the Arrow-Debreu security, which pays 1 if a state occurs and 0 otherwise.

How demanding is the assumption of minimal updating of awareness? Proposition 1 shows that a trader increases her awareness if and only if she becomes aware of something that others are already aware of. Moreover, under a mild condition, she never becomes aware of something that others are not aware of. In other words, agreement and information aggregation does not depend on the ability of traders to increase awareness unboundedly. Instead, they only need to be able to understand each other, by becoming aware of the contingencies that rationalise the actions of others. Such an assumption is significantly weaker than requiring that the information structure is common knowledge, as in an environment with full awareness.

The trading mechanism that we use in this paper is the Market Scoring Rule (MSR), also employed in prediction markets (McKelvey and Page [1990], Hanson [2003, 2007]). It specifies that, in period 0, an uninformed market maker provides an initial announcement, which we interpret as the starting price of security X. In period 1, trader 1 revises the announcement, in period 2 trader 2 makes another announcement, and so on, until period n and trader n. In period $n+1$, trader 1 provides a new announcement and the cycle repeats. When all traders no longer revise their announcement, the process ends and the true value of the security is revealed. We assume that each trader is myopic, so that she only cares about her current period payoff, when making an announcement.

Payoffs in each period are determined by a proper scoring rule (e.g. quadratic rule). They are a function of the true value of the security, the current announcement and that of the previous period. Proper scoring rules ensure that the announcement which maximises a myopic trader's current payoff is the expected value of the security, according to her beliefs. Since each myopic trader announces the expected value of the security given her beliefs, the setting is similar to that of Geanakoplos and Polemarchakis [1982], where agents announce their posterior of an event.

1.1 Related Literature

Starting with Fagin and Halpern [1988], there is a growing literature on unawareness. Foundational models have been developed, among others, by Modica and Rustichini [1994, 1999], Halpern [2001], Li [2009], Halpern and Rêgo [2005, 2008], Heifetz et al. [2006, 2008], Board and Chung [2007] and Galanis [2011a, 2013]. An overview of the literature is provided in Schipper [2015], including several applications with unawareness.

The ability of markets to aggregate information has been analysed at least since Hayek [1945] and Grossman [1976]. Radner [1979] introduced the concept of Rational Expectations Equilibrium (REE) and proved that generically prices aggregate information. There is a large literature on REE and their convergence in dynamic settings (e.g. Hellwig [1982], Nielsen [1984], McKelvey and Page [1986], Dubey et al. [1987], Wolinsky [1990], Nielsen et al. [1990] and Golosov et al. [2014]).

The no-trade theorems stem from the static model Aumann [1976] and the two-period model of Milgrom and Stokey [1982]. Geanakoplos and Polemarchakis [1982] analyze a dynamic version of the no-trade theorem, whereas Cave [1983], Sebenius and Geanakoplos [1983], Nielsen [1984] and Nielsen et al. [1990] generalise using other aggregate statistics.

DeMarzo and Skiadas [1998, 1999] study both fully and partially revealing REE, providing several results on separable securities. Ostrovsky [2012] and Chen et al. [2012] show that separable securities are both necessary and sufficient for information aggregation, irrespective of whether traders are myopic or strategic. Galanis et al. [2019] extend these results to an environment with ambiguity. They use the market scoring rule of McKelvey and Page [1990] and Hanson [2003, 2007], although Ostrovsky [2012] proves the same result also in the framework of Kyle [1985]. Similar approaches can be found in Chen et al. [2010] and Dimitrov and Sami [2008], where more specific signal structures are examined.

Speculative trading behavior in environments with unawareness has been studied mostly with static models (e.g. Galanis [2013, 2018], Heifetz et al. [2013a], Meier and Schipper [2014]). Our setting is multi-period and we explicitly model how information and awareness are updated, when a counterfactual announcement is made. Grant and Quiggin [2013], Rêgo and Halpern [2012], Heifetz et al. [2013b] and Halpern and Rêgo [2014] study dynamic games with differential awareness. Karni and Vierø [2013, 2017] study how awareness is updated and the state space is enlarged in a decision theoretic model.

The idea of endogenously updating awareness, in a minimal way, is also present in Schipper [2018], who studies self-confirming equilibria in games with unawareness. By observing the actions of others, a player may become more aware, which is expressed by having her play a different game. A self-confirming game is reached when no-one is updating her awareness endogenously, due to observing the actions of others.

The paper proceeds as follows. We first present an example in Section 2, in order to illustrate our approach. Section 3 presents the model and the main result is provided in Section 4. Proofs are contained in the Appendix.

2 An example

Before proceeding with the formal model, we first present a simple example of dynamic trading with two agents, who trade an asset that is a derivative on the value of a company.[2] There are three payoff relevant states and two dimensions which determine the value of the company. The first is whether there will be a merger, which occurs only in the first state. The second is whether the CEO has engaged in insider trading, which occurs only in the third state. In the second state, there is no insider trading because the authorities arrest the CEO in time, however this means that there will be no merger either. The asset pays one util in the first state, where the value of the company increases, and zero otherwise.

Trader 1 is fully aware of both these contingencies but she is only informed about

[2] A slightly different and static example is presented in Galanis [2018], where it is shown that an always beneficial bet does not imply no common priors.

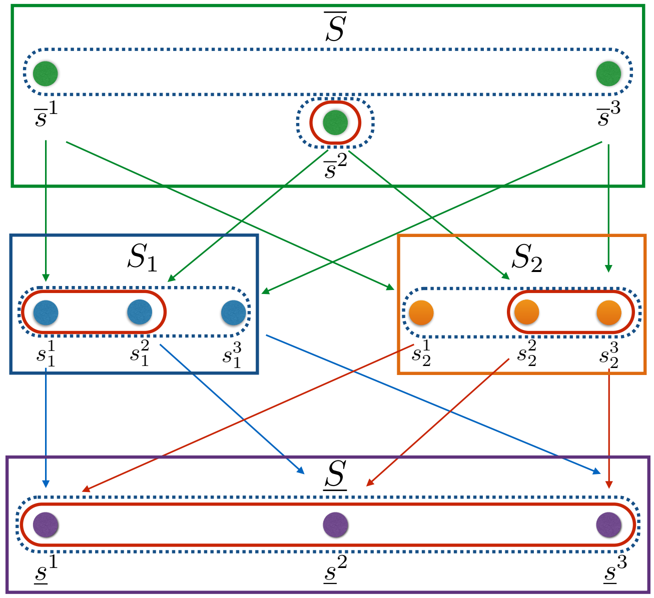

Figure 1: Information structure in periods 0 and 1

whether there is insider trading. Trader 2 is not always fully aware. She becomes aware of the possibility of a merger (but not of insider trading) in the first state. Although she still does not know whether a merger will fail, she is confident that there is no other contingency that would decrease the company's value, making the asset pay 0. Similarly, she becomes aware of insider trading (but not of the merger) in the third state. Although she does not know whether the insider trading will take place, she cannot foresee any good outcome that would increase the company's value, hence she thinks that the asset will pay 0. In the second state, she becomes aware of both contingencies and she also learns that there will be no merger or insider trading, due to the arrest by the authorities. She is fully aware and fully informed that the asset will pay 0. In every state, the traders could learn the value of the asset by pooling their private information. The question we are asking in this paper is whether the value can also be revealed by the fluctuating price of the asset, in a dynamic trading environment.

The information structure of the traders is depicted in Figure 1. Different levels of awareness are expressed by having disjoint state spaces and in this example there are four, $\underline{S}$, S_1, S_2 and $\overline{S}$. State space $\overline{S}$ expresses the highest level of awareness $\underline{S}$ the lowest, which we write as $\overline{S} \succeq S_1, S_2 \succeq \underline{S}$. State spaces S_1 and S_2 are not comparable in terms of their level of awareness, because the former (latter) expresses the possibility of a merger (insider trading) only. Each state space S has three states, depicted as dots in Figure 1. By saying that a state $s \in S$ projects to state $s' \in S'$, where $S' \succeq S$, we mean that a trader who is aware of S' but not S only perceives the limited expressiveness of s', rather than that of s. The projections in Figure 1 are given by the thin arrows, so that for $k = 1, 2, 3$, state $\overline{s}^k$ projects to s_1^k and to s_2^k, which in turn both project to $\underline{s}^k$.

The two traders share a common prior, which on each state space is $(1/3, 1/3, 1/3)$. Both traders are always aware of the bottom, or payoff relevant, state space $\underline{S}$. They trade an Arrow-Debreu security X which pays 1 if $\underline{s}^1$ occurs and 0 if either $\underline{s}^2$ or $\underline{s}^3$ occur. Because the two traders are always aware of $\underline{S}$, where X is defined, we do not model unawareness of payoff relevant events.

Trader 1's information structure in period 0 specifies that, on the full state space $\overline{S}$, she has a partition, so that $P^1(s_3^1) = P^1(s_3^3) = \{s_3^1, s_3^3\}$ and $P^1(s_3^2) = \{s_3^2\}$. All other state spaces specify that she is completely uninformed, so that $P^1(s_k^j) = S_k$ and $P^1(\underline{s}^j) = \underline{S}$, where $k = 1, 2$, $j = 1, 2, 3$. Her information structure is depicted in Figure 1 by the discontinuous enclosures. Trader 2's information structure in period 0, depicted by the solid enclosures, specifies that $P^2(\overline{s}^1) = \{s_1^1, s_1^2\} = P^2(s_1^1) = P^2(s_1^2), P^2(\overline{s}^2) = \{\overline{s}^2\}, P^2(\overline{s}^3) = \{s_2^2, s_2^3\} = P^2(s_2^2) = P^2(s_2^3), P^2(s_1^3) = P^2(s_2^1) = P^2(\underline{s}^j) = \underline{S}$, where $j = 1, 2, 3$.

In words, the full state space $\overline{S}$ describes that trader 2 is fully aware and has perfect information at $\overline{s}^2$, whereas at $\overline{s}^1$ she is only aware of S_1 (merger) and considers s_1^1 and s_1^2 to be possible. At state $\overline{s}^3$, she is only aware of S_2 (insider trading) and considers s_2^2 and s_2^3 to be possible. On the other hand, the full state space describes that trader 1 is always fully aware and knows whether $\overline{s}^2$ has occurred or not.[3]

Although the two traders might have different awareness across states, what matters for payoffs is only state space $\underline{S}$. For example, when trader 1 knows at $\overline{s}^2$ that state $\overline{s}^2$ occurred, she also knows that state s_0^2 occurred, hence the security X pays 0. Note that if the two traders could speak with each other and reveal the information they have about the payoff relevant state space $\underline{S}$, they would collectively always know which state is true and therefore the true value of the security. Would the same occur if, instead of pooling their information, they traded security X in a dynamic setting? In other words, do prices aggregate information?

To study this question, consider the following trading procedure, which is called the Market Scoring Rule (McKelvey and Page [1990], Hanson [2003, 2007]). Suppose that the true state is s_3^1. The market starts in period 0, where a market maker who is only aware of the payoff relevant $\underline{S}$ and has no private information, posts an initial price of 1/3, which is the expected value of the asset. Since the market maker is uninformed and unaware, her announcement does not trigger any updating of information or awareness. The information structure in period 1 is the same as that of period 0.

In period 1, it is trader 1's turn to make an announcement. Her payoff from period 1 is

$$-(y_1 - x^*)^2 + (y_0 - x^*)^2,$$

where y_1 is her announcement, $y_0 = 1/3$ is the previous announcement and $x^* = 1$ is the true value of the security. Since trader 1 does not know x^*, she chooses an announcement that maximises the expected value of the expression, given her posterior beliefs. We are effectively using the quadratic rule, so that the market maker's "score"

[3] Note that if we were to "flatten" the information structure, by projecting all information sets to the lowest state space $\underline{S}$, we would obtain a non-partitional information structure, like the one studied in Geanakoplos [1989]. Its connection with the value of information in environments with unawareness has been studied by Galanis [2015, 2016], among others.

by announcing y_0 is $-(y_0 - x^*)^2$. Trader 1's score from announcing y_1 is $-(y_1 - x^*)^2$. Her payoff is just the expected difference of these two scores.

The quadratic rule is a special case of a proper scoring rule. Its defining property is that for every posterior belief, the expected value of the difference is maximised by announcing the expected value of X. We explain these concepts in more detail in Section 3.2.

In period 1 and at s_3^1, trader 1's posterior beliefs over the payoff relevant state space $\underline{S}$ are $(1/2, 0, 1/2)$. Because trader 1 is myopic, she only cares about her period 1 payoff, hence she announces the expected value of X, which is $1/2$. This is equivalent to buying the security from the market maker, who posted an initial price of $1/3$. To see that $1/2$ is indeed the solution, note that the second part of the payoff does not depend on y_1, hence trader 1 chooses $y_1 \in [0, 1]$ that maximises $-1/2(y_1 - 1)^2 - 1/2(y_1 - 0)^2$.

Prices reveal information. Trader 2 understands that only states which describe that trader 1's announcement would be $1/2$, are possible. In an environment without unawareness, she would always consider such states to be possible, because the information structure is common knowledge. In an environment with unawareness, however, this is not the case, because low awareness might mean that trader 2 has a wrong understanding about 1's information structure. Indeed, at s_3^1 trader 2 is aware of S_1 and considers s_1^1, s_1^2 to be possible. Both these states describe that trader 1 is only aware of the payoff relevant state space $\underline{S}$ and has no information. Hence, she should announce $1/3$, instead of $1/2$.

Hearing the counterfactual announcement of $1/2$ is totally surprising for trader 2. How should she react? One possibility is that she ignores any public information that contradicts her own private information, either because she does not understand it, or because she thinks it is wrong. In other words, she behaves like a noise trader who ignores what others are doing. In such a case, we cannot expect that there can be agreement or information aggregation.

The other possibility is that trader 2 tries hard to rationalise the counterfactual announcement and, in the process, increases her awareness. In this example, the only state space which is more expressive than S_2 is $\overline{S}$. In the model, we assume that traders are always able to update their awareness in a minimal way.

When trader 2 updates her awareness to $\overline{S}$ in period 2, she reinterprets the public information revealed by announcement y_1. In particular, she understands that an announcement of $1/2$ could arise if the state is either $\overline{s}^1$ or $\overline{s}^3$. We denote by $F_1^y(\overline{S}) = \{\overline{s}^1, \overline{s}^3\}$ the public information revealed by announcements up to period 1 and expressed in state space $\overline{S}$. Note that $F_1^y(S_1) = \emptyset$, because S_1 cannot explain an announcement of $1/2$. Hence, trader 2 needs to increase her awareness to $\overline{S}$.

Trader 2's private information in period 2, $P_2^2(s_3^1)$, is the conjunction of two events. The first is her period 0 private information, $P_0^2(s_3^1) = \{s_1^1, s_1^2\}$, enlarged to her awareness in period 2, $\overline{S}$, so that $(P_0^2(s_3^1))^{\overline{S}} = \{s_3^1, s_3^2\}$. The second is the public information expressed by $\overline{S}$, $F_1^y(\overline{S}) = \{\overline{s}^1, \overline{s}^3\}$. Hence, trader 2 knows that the true state is s_3^1.

The new information structure in period 2 is depicted in Figure 2. State spaces S_1 and S_2 cannot rationalise announcement y_1, hence they are dropped. Although the same is true for $\underline{S}$, we keep it for convenience, as the payoffs of X are defined there. At s_3^1, trader 2 knows that the true state is s_3^1, however trader 1 considers both s_3^1 and

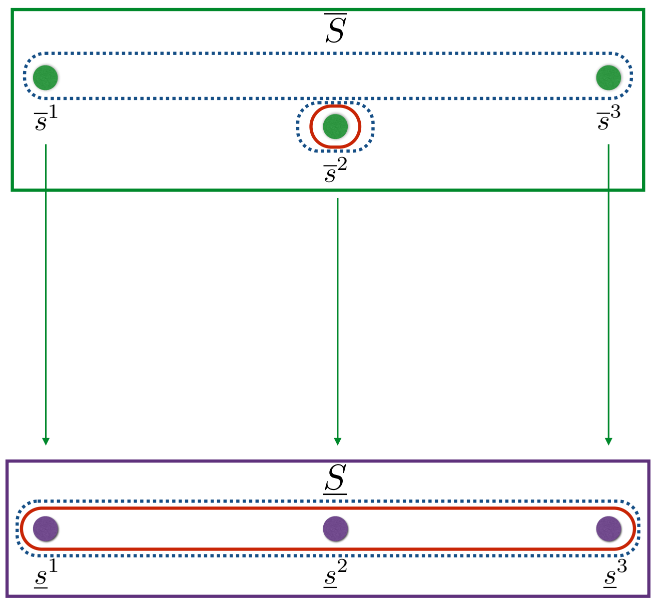

Figure 2: Information structure in period 2

s_3^2 to be possible.

In period 2, trader 2's posterior beliefs on the full state space are $(1, 0, 0)$ and she announces 1, which is the expected value of X. This announcement reveals to trader 1 that the true state is s_3^1, hence in period 3 she also announces 1.

The two traders have reached an agreement on the price of the security. More importantly, they agree on the correct price of the security, hence there is information aggregation. Theorem 1 shows that this is always the case, as long as traders are always able to minimally update their awareness, after listening to a counterfactual announcement.

3 The Model

3.1 Preliminaries

The model is a reduced version of Galanis [2013], which is based on Heifetz et al. [2006]. The main difference from the latter model is that we do not impose the Projection Preserve Knowledge property, so that lower awareness may imply a wrong view of the information of others.[4]

We first present the static model and then add a time dimension. There are I traders with $|I| = n$. Different levels of awareness are represented by disjoint state spaces. Let $\mathcal{S} = \{S_a\}_{a \in A}$ be the finite collection of all these state spaces. We assume

[4]For a comparison with Heifetz et al. [2006] and Li [2009], see Galanis [2013] and its syntactic version Galanis [2011a].

that $\mathcal{S}$ is a complete lattice with a partial order $\preceq$.[5] If $S \preceq S'$, we say that S' is (weakly) more expressive than S or, equivalently, that a trader whose state space is S' is more aware than a trader whose state space is S. By construction, there is a top, or full state space $\overline{S}$, and a bottom, or payoff relevant, state space $\underline{S}$. That is, for all $S \in \mathcal{S}$, $\underline{S} \preceq S \preceq \overline{S}$.

A state s is an element of some state space S. Let $\Sigma = \bigcup_{S \in \mathcal{S}} S$ be the set of all states. We assume that every state space $S \in \mathcal{S}$ has finitely many states. An *event* E is a subset of some state space $S \in \mathcal{S}$.

If $S \preceq S'$, so that S' is more expressive than S, we require that each state $s' \in S'$ can be mapped to its "restricted" image in the less expressive S. Formally, we require that there is a surjective projection $r_S^{S'} : S' \to S$. Projections are required to commute: if $S \preceq S' \preceq S''$, then $r_S^{S''} = r_S^{S'} \circ r_{S'}^{S''}$.

Let $E \subseteq S, E' \subseteq S'$ be two events and suppose $S'' \succeq S' \succeq S$. We denote the projection of set E' to the less expressive S by $E'_S = \bigcup \{ r_S^{S'}(s') \in S : s' \in E' \}$. We denote the enlargement of $E' \preceq S''$ to the more expressive S'' by $E'^{S''} = \bigcup \{ s'' \in S'' : r_{S'}^{S''}(s'') \in E' \}$.

Trader i's information structure is represented by a possibility correspondence $P^i : \Sigma \to 2^{\Sigma} \setminus \emptyset$. The interpretation is that at $s \in S$, i considers $P^i(s)$ to be possible. We assume that P^i has the following properties:

(0) Confinement: If $s \in S$ then $P^i(s) \subseteq S'$ for some $S' \preceq S$.

(1) Generalized Reflexivity: $s \in (P^i(s))^{\uparrow}$ for every $s \in \Sigma$.

(2) Stationarity: $s' \in P^i(s)$ implies $P^i(s') = P^i(s)$.

(3) Projections Preserve Ignorance: If $s \in S$ and $S' \preceq S$ then $(P^i(s))^{\uparrow} \subseteq (P^i(s_{S'}))^{\uparrow}$.

(4) Projections Preserve Awareness: If $s \in S$, $s \in P^i(s)$ and $S' \preceq S$ then $s_{S'} \in P^i(s_{S'})$.

These properties are discussed extensively in Heifetz et al. [2006] and Galanis [2013]. Let $S^i(s)$ denote trader i's state space at $s \in \Sigma$. In particular, $S^i : \Sigma \to \mathcal{S}$ is such that for any $s \in \Sigma$, $S^i(s) = S$ if $P^i(s) \subseteq S$. If $S^i(s) \succ S^j(s)$ then we say that *trader i is more aware than trader j at s.*

Because there are many state spaces, we define a generalized prior π on the full state space $\overline{S}$ and we assume that the prior for a trader who is aware of a lower state space S is just the marginal of π on S. Formally, a generalized prior is a function $\pi : 2^{\Sigma} \setminus \emptyset \to [0, 1]$ such that the restriction of π on $\overline{S}$ is a probability distribution and, for any nonempty event $E \subseteq S$, $\pi(E) = \pi(E^{\overline{S}})$. We assume a strictly positive generalised prior, so that $\pi(s) > 0$ for all $s \in \Sigma$.

A trader updates her beliefs using Bayes' rule, given her information and awareness. In particular, if her information is $P^i(s)$, she assigns probability $\frac{\pi(s')}{\pi(P^i(s'))}$ to state s' if $s' \in P^i(s)$ and 0 otherwise.

Definition 1. *A (static) unawareness structure is a tuple $\mathbb{U} = \langle \mathcal{S}, \{ r_{S^{\beta}}^{S^{\alpha}} \}_{S^{\beta} \preceq S^{\alpha}}, \{ P^i \}_{i \in I}, \pi \rangle$, where the collection of state spaces $\mathcal{S}$ and each state space $S \in \mathcal{S}$ are finite.*

[5] A complete lattice is a partially ordered set in which all subsets $\mathcal{G} \subseteq \mathcal{S}$ have both a supremum (or join, denoted $\bigvee \mathcal{G}$) and an infimum (or meet, denoted $\bigwedge \mathcal{G}$).

3.2 Trading environment

Traders share a common, strictly positive, generalized prior π. We assume that for each $s^* \in \overline{S}$, $\bigcap_{i \in I} P^i(s^*)_{\underline{S}} \in \underline{S}$ is a unique element of the payoff relevant state space $\underline{S}$, that everyone is always aware of. This means that if the traders could truthfully pool their information about $\underline{S}$, they would always learn the true payoff relevant state.

The market mechanism is based on Ostrovsky [2012]. There are infinitely many periods $t = 0, 1, \ldots$, where traders buy and sell security $X : \Sigma \to \mathbb{R}$, which pays according to the state in Σ. We assume that $X(s) = X(\underline{s})$ for all $\underline{s} \in \underline{S}$ and $s \in (r^S_{\underline{S}})^{-1}(\underline{s})$. This implies that payoffs are determined only by the bottom state space $\underline{S}$, that all traders are always aware of. In other words, traders are never unaware of payoff relevant events in $\underline{S}$. However, they may be unaware of events in other state spaces, whose occurrence is correlated with the occurrence of payoff relevant events in $\underline{S}$.

At each time t, the relevant unawareness structure is denoted by $\mathbb{U}_t$. At time t_0, a state $s^* \in \overline{S}$ is drawn using the generalized prior π. The realised payoff relevant state is the projection of s^* to $\underline{S}$, $\underline{s} = \{s^*\}_{\underline{S}}$. Trading starts in period 0, when the uninformed market maker posts the initial price $y_0 \in [\underline{y}, \overline{y}]$ of security X, where $\underline{y} = \min_{s \in \underline{S}} X(s)$, $\overline{y} = \max_{s \in \underline{S}} X(s)$. At time t_1, trader 1 makes an announcement $y_1 \in [\underline{y}, \overline{y}]$, at t_2 trader 2 makes an announcement $y_2 \in [\underline{y}, \overline{y}]$, and so on. At time t_{n+1}, trader 1 makes an announcement again. There are infinitely many rounds of announcements. However, since the state space is finite and each trader changes her announcement only if her information changes, after some period t each trader stops changing her announcement. In principle, these final announcements can be different among traders.

When everyone stops updating their announcements, the true value of the security, $X(s^*) = x^*$, is revealed and the payoffs for all traders are calculated using a market scoring rule (MSR), which is based on a proper scoring rule r.

A scoring rule is a function $r(y, x^*)$, where y is an announcement and x^* is a realization of a random variable. It is proper if, for any random variable X and any probability measure π, the expectation of r is maximized at $y = E_\pi(X)$. It is strictly proper if $y = E_\pi(X)$ is the unique maximizer. This means that a trader who only cares about maximising the score r, should announce the expected value of X according to her own beliefs. An example of a proper scoring rule is the quadratic rule, $r(y, x^*) = -(y - x^*)^2$, that we used in the example of Section 2.

The MSR leverages the "truth-telling" property of proper scoring rules in a market setting. It specifies that if trader i makes announcement y_k at time t_k, the previous trader announced y_{k-1} and the true value of the security is x^*, then trader i's payoff in that round is

$$r(y_k, x^*) - r(y_{k-1}, x^*).$$

Note that if i repeats the previous announcement, so that $y_k = y_{k-1}$, then the payoff is 0. However, it can also be negative or positive.

Throughout the paper we assume that each trader is not strategic but myopic, so that she only cares about the current period's payoff and maximizes the expected value of $r(y_k, x^*) - r(y_{k-1}, x^*)$, given her posterior beliefs. Because $r(y, x^*)$ is a proper scoring rule, the announcement is the expected value of X.

We say that information is aggregated if the announcements always converge to the true value of the security.

Definition 2. *Information is aggregated if sequence $y = \{y_t\}_{t\geq 0}$ of announcements converges in probability to $X(s^*)$, for all $s^* \in \overline{S}$.*

The convergence is with respect to the probability distribution on $\overline{S}$, that is implied by the generalized prior π. Since the state space is finite, information aggregation is equivalent to requiring that for each $s^* \in \overline{S}$, there exists some k', such that for all $k > k'$, $y_k = X(s^*)$.

Ostrovsky [2012] shows that in an environment without unawareness and expected utility, separable securities always aggregate information. A security is separable given an information structure if, for every non-degenerate prior on the values of the security, there is a trader who receives an informative signal with positive probability.[6] The most well-known example is the Arrow-Debreu security, which pays 1 if a state occurs and 0 otherwise.

We adapt his definition of separability in the current setting.

Definition 3. *Security X is non-separable under unawareness structure $\mathbb{U}$ if there exist generalized prior π' and value $v \in \mathbb{R}$ such that:*

(i) $\pi'(s^*)$ is positive on at least one state $s^* \in \overline{S}$ in which $X(s^*) \neq v$,

(ii) For every trader i and every full state $s^* \in \overline{S}$ with $\pi'(s^*) > 0$,

$$E_{\pi'}(X|P^i(s^*)) \equiv \frac{\sum\limits_{s' \in P^i(s^*)} \pi'(s')X(s')}{\pi'(P^i(s^*))} = v.$$

Otherwise, security X is separable.

A security X is non-separable under an unawareness structure $\mathbb{U}$, if it is possible to find a generalized prior π' such that for all full states in the support of π', all traders make the same announcement v, so that there is agreement.[7] Yet, X does not always pay v, so there is no information aggregation. Note that π' may be different from the generalized prior π of $\mathbb{U}$.

3.3 Public information

Announcement y_k about the price of security X reveals some of the private information of the trader who made it. In an environment without unawareness, this announcement

[6] The intuition behind the term "separable" can be found in Theorem 7 of Ostrovsky [2012], which characterises separable securities using the following condition. For every value $v \in \mathbb{R}$, we can find multipliers $\lambda_i(P^i(s)) \in \mathbb{R}$, one for each state s and trader i, such that for each state s with $X(s) \neq v$, the sign of $X(s) - v$ is the same as the sign of the sum of the multipliers for those partition elements containing s. In other words, value v separates state space S into two regions, one containing states where $X(s)$ is bigger than v and another where it is smaller. If the security is separable, then we can always find multipliers such that the sign of their sum also separates S in the same two regions.

[7] Note that if there is agreement, then traders will no longer update their announcements.

creates public information that everyone interprets in the same way, as there is only one state space. With unawareness, however, each state space S expresses a possibly different public information.

Consider a sequence of announcements $y = \{y_t\}_{t\geq 0}$, where y_0 is the initial announcement of the uninformed market maker. Let $F_0^y(S) = S$ be the public information revealed by y_0, expressed in state space S. Since the market maker is uninformed, her announcement does not reveal any information.

At time t_1, let $F_1^y(S) = \{s \in F_0^y(S) : E_\pi(X|P_t^1(s)) = y_1\}$ be the public information revealed by trader 1's announcement, expressed in state space S.[8] It contains all states in $F_0^y(S)$ which describe that 1's conditional expectation of security X is equal to the actual announcement y_1. In general, $F_1^y(S)$ could be the empty set, if S is unable to rationalise announcement y_1, as we showed in Section 2 with state space S_1.

Denote by $i(t)$ the trader who makes an announcement at time t. At time $t \geq 1$, if $F_{t-1}^y(S) \neq \emptyset$, let $F_t^y(S) = \{s \in F_{t-1}^y(S) : E_\pi(X|P_t^{i(t)}(s)) = y_t\}$ be the public information created by all announcements up to time t, expressed in state space S. We can interpret $F_t^y(S)$ as the information of an outside observer, who has no initial private information and her state space (and highest awareness) is S, after observing all announcements up to time t.

3.4 Dynamic awareness and updating

A dynamic unawareness structure $\{\mathbb{U}_t\}_{t\geq 0}$ consists of a sequence of static unawareness structures, $\mathbb{U}_t$, one for each time t.

Definition 4. *$\{\mathbb{U}_t\}_{t\geq 0}$ is a dynamic unawareness structure if, for all $s^* \in \overline{S}$ and all $t \geq 0$, $\mathbb{U}_t = \langle \mathcal{S}_t, \{r_{tS^\beta}^{S^\alpha}\}_{S_t^\beta \preceq_t S_t^\alpha}, \{P_t^i\}_{i\in I}, \pi\rangle$ is a static unawareness structure and $y = \{y_t\}_{t\geq 0}$ is the sequence of announcements with the following properties, for all $s \in S_t \cap S_{t+1}$ and all $i \in I$:*

1. *$\mathcal{S}_{t+1}$ is a subset of $\mathcal{S}_t$ and $\preceq_{t+1}$ is the projection of $\preceq_t$ on $\mathcal{S}_{t+1}$,*
2. *If $S^\alpha, S^\beta \in \mathcal{S}_{t+1}$, then $r_{t+1S^\beta}^{S^\alpha} = r_{tS^\beta}^{S^\alpha}$,*
3. *$S_{t+1}^i(s) \succeq S_t^i(s)$,*
4. *$y_{t+1} = E_\pi(X|P_{t+1}^{i(t+1)}(s^*))$,*
5. *$P_{t+1}^i(s) = (P_0^i(s))^{S_{t+1}^i(s)} \bigcap F_t^y(S_{t+1}^i(s))$,*
6. *If $S_{t+1}^i(s) \succ S_t^i(s)$, then there does not exist $S' \in \mathcal{S}_{t+1}$ with $S_t^i(s) \preceq S' \prec S_{t+1}^i(s)$, such that $(P_0^i(s))^{S'} \bigcap F_t^y(S') \neq \emptyset$.*

The first two conditions specify that each $\mathbb{U}_t$ is a subset of $\mathbb{U}_{t-1}$, so that the collection of state spaces $\mathcal{S}_t$ is a subset of $\mathcal{S}_{t-1}$ and the partial order $\preceq$ and projections between state spaces are preserved. The third condition specifies that each trader cannot be less aware over time. The fourth condition determines that each announcement

[8] Since each trader is myopic, the MSR implies that she will announce the expected value of X, according to her beliefs.

is the expected value of security X, according to the announcer's updated private information. This is implied by our assumptions that each trader is myopic, so she only cares about the current payoff, and the payoffs are determined by the MSR. Announcements are therefore uniquely defined.

The last two properties specify how information and awareness are updated, given a history of announcements. Recall that in period t, each trader knows the history of announcements $y = \{y_t\}_{t \geq 0}$ up to t and can incorporate the public information revealed by them, so that she can update her own private information. In an environment without unawareness, updating is simple. For each $s \in S$, $P^i_{t+1}(s) = P^i_t(s) \cap F^y_t(S)$. This is equivalent to requiring that $P^i_{t+1}(s) = P^i_0(s) \cap F^y_t(S)$. In other words, the private information of i at $t+1$ is the conjunction of i's private information at time 0 and the public information revealed by the announcements up to, and including, t.

In the current setting, however, we also need to specify how awareness is updated. Even though a trader in our model is not aware of what she is unaware of, she may realise that she lacks awareness in order to understand the history of announcements. This occurs only when she hears a counterfactual announcement, so that there is no state that she considers possible and can rationalise it, because $P^i_{t+1}(s) = P^i_t(s) \cap F^y_t(S) = \emptyset$.

When the announcement is not counterfactual, she just updates her information. If it cannot be rationalised, however, then there are two possibilities. First, the trader realises that she misses something, however she cannot update her awareness. As she ignores any future announcements or the information that they reveal, she behaves like a noise trader. In such a case, we cannot expect that there will be agreement or information aggregation.[9] Definition 4 explicitly excludes such a possibility.

The second possibility is that each trader can update her awareness when she hears a counterfactual announcement that she could not rationalise with her existing awareness. We require that such updating is minimal. It is important to note that, relative to the dynamic unawareness structure, each trader i's state space $S^i_t(s)$ is uniquely determined by the announcement and the possibility set $P^i_t(s)$.

Property 5 of Definition 4 decomposes the trader's private information at $t+1$ into her private information at initial time 0 and the public information at t, given all announcements up to time t. If there is no updated awareness from t to $t+1$, so that $S^i_{t+1}(s) = S^i_t(s)$, this condition reduces to $P^i_{t+1}(s) = P^i_0(s) \cap F^y_t(S^i_{t+1}(s))$, just like the case of no unawareness. With unawareness, i's private information at 0 is enlarged to i's state space at $t+1$, $S^i_{t+1}(s)$, and the public information is interpreted within the same state space.

Property 6 specifies that updating of awareness is minimal. This means that the trader gains new awareness only if her previous period awareness was not enough to explain the t announcement, because $(P^i_0(s))^{S^i_t(s)} \bigcap F^y_t(S^i_t(s)) = \emptyset$. Moreover, it is not possible to explain the announcement with a state space S' which is less expressive

[9]Note that, in general, it is also possible that a trader excludes the projection of the true state to her state space, s^*_S. This can happen because she might not have a correct understanding of the information structure of other traders. We explicitly exclude such a case by requiring that each $\mathbb{U}_t$ is an unawareness structure, satisfying Generalized Reflexivity. If we allow for the violation of Generalized Reflexivity, then information aggregation may not occur.

than $S^i_t(s)$ and more expressive than $S^i_{t+1}(s)$.

The following Proposition identifies two important properties of the updating process. The first specifies that a trader updates her awareness if and only if she becomes aware of something that the trader who made the announcement is already aware of. This is the *only* way of rationalizing the announcement. Effectively, our updating process ensures that traders increase their awareness only by working out what the others are aware of. Agreement therefore only requires that traders eventually understand each other, rather than always being fully aware.

Whereas the first property places a lower bound on the updating of awareness, the second specifies an upper bound. In particular, if j makes an announcement at time $t-1$ and i increases her awareness from $t-1$ to t, the join of their awareness from $t-1$ to t does not change. In other words, trader i can at most become aware of something that j is already aware of. This implies that agreement does not require that the collective awareness of the group of traders increases. For this result, we require that state space $S' = \Big((S^i_t(s^*) \wedge S^j_{t-1}(s^*)\Big) \vee S^i_{t-1}(s^*)$ can rationalise all announcements up to $t-1$, so that $F^y_{t-1}(S') \neq \emptyset$. The first part in parenthesis is the meet of i's awareness at t and j's awareness at $t-1$. This is the new awareness that i needs in order to rationalise j's last announcement. The second part is $S^i_{t-1}(s^*)$, i's awareness at $t-1$.

Proposition 1. *Fix a dynamic unawareness structure $\{\mathbb{U}_t\}_{t\geq 0}$ and suppose that trader j makes an announcement in period $t-1$. Then, for any full state $s^* \in \overline{S}$,*

- *Trader $i \neq j$ updates her awareness at t, so that $S^i_{t-1}(s^*) \prec S^i_t(s^*)$, if and only if*

$$S^i_{t-1}(s^*) \wedge S^j_{t-1}(s^*) \prec S^i_t(s^*) \wedge S^j_{t-1}(s^*),$$

- *If $F^y_{t-1}\left(\Big((S^i_t(s^*) \wedge S^j_{t-1}(s^*)\Big) \vee S^i_{t-1}(s^*)\right) \neq \emptyset$, then*

$$S^i_t(s^*) \vee S^j_t(s^*) = S^i_{t-1}(s^*) \vee S^j_{t-1}(s^*).$$

4 Agreement and information aggregation

The updating process described in the previous section specifies that the traders are always able to increase their awareness when they hear an announcement that contradicts their information. The awareness update is minimal, in the sense that the new state space is the least expressive that can rationalise the history of announcements up to the current period. Moreover, each trader never excludes the projection of the true state on her state space, so Generalized Reflexivity is always satisfied.

If one of these (strong) conditions are not met, then at least one trader may not be able to incorporate the correct public information and will not change her private information or her announcement. This implies that traders may not reach an agreement on the price of the security, or they may agree on the wrong price, hence no information aggregation.

The first result of Theorem 1 below shows that when awareness updating is minimal and (a restricted view of the) truth is never excluded, traders eventually reach an agreement on the price of the security. This is a generalisation of the result of Geanakoplos and Polemarchakis [1982], in an environment without unawareness. The second result shows that if security X is separable under the "last" unawareness structure, which is generated when agreement has been reached, then traders agree on the price which is equal to the value of X at the true state, hence there is also information aggregation. This is a generalization of Ostrovsky [2012] for the non-strategic environment without unawareness. One difference with Ostrovsky [2012] is that the condition of separability is imposed on the "last" unawareness structure, not the the first. The reason is that separability is preserved in the standard setting, under Bayesian updating and no awareness updating. However, when there is also awareness updating, this may not be the case.

Theorem 1. *Fix a dynamic unawareness structure $\{\mathbb{U}_t\}_{t\geq 0}$. For any full state $s^* \in \overline{S}$,*

- *There exists a finite $\bar{t}$ such that $y_t = y_{\bar{t}}$ for all $t \geq \bar{t}$,*
- *If security X is separable under $\mathbb{U}_{\bar{t}}$, then there is information aggregation.*

As we have already mentioned, minimal awareness updating and Generalized Reflexivity are strong conditions. If a trader cannot update her awareness when hearing a counterfactual announcement, then she may keep repeating her current announcement, therefore not achieving agreement. Moreover, if Generalized Reflexivity is violated, then traders might agree on the wrong price of the security, hence there will be no information aggregation. On the other hand, we can think of the current environment as imposing weaker conditions than the standard one which assumes full awareness for everyone and common knowledge of the information structure.

Finally, the result of information aggregation does not depend on the specific order with which traders make their announcements. That is, changing the order of traders would not influence the final announcements, which would still be equal to the true value of the security, for each full state $s^* \in \overline{S}$. However, the intermediate announcements and the dynamic unawareness structure could be different.

A Appendix

Proof of Proposition 1. For the first claim, let $j \neq i$ be the trader who makes the announcement at $t-1$. By definition, $S^i_{t-1}(s^*) \wedge S^j_{t-1}(s^*) \prec S^i_t(s^*) \wedge S^j_{t-1}(s^*)$ implies $S^i_{t-1}(s^*) \prec S^i_t(s^*)$ and i updates her awareness at t.

Conversely, suppose that at t trader i updates her awareness, so that $S^i_{t-1}(s^*) \prec S^i_t(s^*) \equiv S$, but it is not the case that $S^i_{t-1}(s^*) \wedge S^j_{t-1}(s^*) \prec S^i_t(s^*) \wedge S^j_{t-1}(s^*)$. Note that $S^i_{t-1}(s^*) \preceq S^i_t(s^*)$ implies $S^i_{t-1}(s^*) \wedge S^j_{t-1}(s^*) \preceq S^i_t(s^*) \wedge S^j_{t-1}(s^*)$. By the definition of a lattice, if $A \preceq B$ but not $A \prec B$, then $A = B$. We therefore have $S^i_{t-1}(s^*) \wedge S^j_{t-1}(s^*) = S^i_t(s^*) \wedge S^j_{t-1}(s^*) = S \wedge S^j_{t-1}(s^*) \equiv S'$.

We next show that $P^j_{t-1}(s^*_S) = P^j_{t-1}(s^*_{S'})$. Since $S' \preceq S$ and from Projections Preserve Ignorance, we have $S^j_{t-1}(s^*_S) \succeq S^j_{t-1}(s^*_{S'})$. Also, $S^j_{t-1}(s^*) \succeq S^j_{t-1}(s^*_S)$ and $S \succeq S^j_{t-1}(s^*_S)$ imply $S' = S^j_{t-1}(s^*) \wedge S \succeq S^j_{t-1}(s^*_S) \wedge S = S^j_{t-1}(s^*_S)$. Moreover, $S \wedge S^j_{t-1}(s^*) = S'$ implies that $S^j_{t-1}(s^*) \succeq S'$. Projections Preserve Awareness implies that

$S^j_{t-1}(s^*_{S'}) = S'$. However, $S^j_{t-1}(s^*_{S'}) = S' \succeq S^j_{t-1}(s^*_S)$, therefore $S^j_{t-1}(s^*_{S'}) = S' = S^j_{t-1}(s^*_S)$. Stationarity and $S^j_{t-1}(s^*_S) = S^j_{t-1}(s^*_{S'}) = S'$ imply $P^j_{t-1}(s^*_S) = P^j_{t-1}(s^*_{S'})$.

From Generalized Reflexivity, we have that $s^*_{S^i_{t-1}(s^*)} \in P^i_{t-1}(s^*) = P^i_0(s^*)^{S^i_{t-1}(s^*)} \cap F^y_{t-2}(S^i_{t-1}(s^*))$ and $s^*_{S^i_t(s^*)} \in P^i_t(s^*) = P^i_0(s^*)^{S^i_t(s^*)} \cap F^y_{t-1}(S^i_t(s^*))$. Moreover, i updates her awareness from $t-1$ to t because $(P^i_0(s^*))^{S^i_{t-1}(s^*)} \bigcap F^y_{t-1}(S^i_{t-1}(s^*)) = \emptyset$.

Because $S \succeq S^i_{t-1}(s^*) \succeq S'$ we have $S^j_{t-1}(s^*_S) \succeq S^j_{t-1}(s^*_{S^i_{t-1}(s^*)}) \succeq S^j_{t-1}(s^*_{S'})$. Hence, $S^j_{t-1}(s^*_S) = S^j_{t-1}(s^*_{S^i_{t-1}(s^*)}) = S^j_{t-1}(s^*_{S'})$ and $P^j_{t-1}(s^*_S) = P^j_{t-1}(s^*_{S^i_{t-1}(s^*)}) = P^j_{t-1}(s^*_{S'})$. This implies that both $s^*_{S^i_{t-1}(s^*)}$ and $s^*_{S^i_t(s^*)}$ describe the same information and awareness about j. Since $s^*_{S^i_t(s^*)} \in F^y_{t-1}(S^i_t(s^*))$, it means that $s^*_{S^i_t(s^*)}$ can rationalise announcement y_{t-1}. But then this means that $s^*_{S^i_{t-1}(s^*)}$ can also rationalise y_{t-1}, so that $s^*_{S^i_{t-1}(s^*)} \in F^y_{t-1}(S^i_{t-1}(s^*))$, contradicting that $P^i_0(s^*)^{S^i_{t-1}(s^*)} \cap F^y_{t-1}(S^i_{t-1}(s^*)) = \emptyset$.

For the second claim, by construction j does not update her awareness at t, as she is the one making the announcement at $t-1$. If i also does not update her awareness, then the result is true. Suppose now that at time t, trader i updates her awareness, so that $S^i_{t-1}(s^*) \prec S^i_t(s^*) \equiv S$, but $S^i_{t-1}(s^*) \vee S^j_{t-1}(s^*) \prec S^i_t(s^*) \vee S^j_t(s^*)$. We will show that there exists S' such that $S^i_{t-1}(s^*) \preceq S' \preceq S$, $s^*_{S'}$ rationalizes j's announcement and $S^i_{t-1}(s^*) \vee S^j_{t-1}(s^*) = S' \vee S^j_{t-1}(s^*)$. These three conditions imply that i updating her awareness to $S^i_t(s^*)$ is not minimal.

Define $S' \equiv (S \wedge S^j_{t-1}(s^*)) \vee S^i_{t-1}(s^*)$. Since $S \succeq (S \wedge S^j_{t-1}(s^*))$ and $S \succeq S^i_{t-1}(s^*)$, we have $S \succeq S'$. It is also straightforward that $S^i_{t-1}(s^*) \preceq S'$. From Lemma 6.1 in Davey and Priestley [1990] we have $S' = (S \wedge S^j_{t-1}(s^*)) \vee S^i_{t-1}(s^*) \preceq (S \vee S^i_{t-1}(s^*)) \wedge (S^j_{t-1}(s^*) \vee S^i_{t-1}(s^*)) = S \wedge (S^j_{t-1}(s^*) \vee S^i_{t-1}(s^*)) \preceq S^j_{t-1}(s^*) \vee S^i_{t-1}(s^*)$. Hence, $S' \vee S^j_{t-1}(s^*) \preceq S^i_{t-1}(s^*) \vee S^j_{t-1}(s^*)$. It is straightforward that $S' \vee S^j_{t-1}(s^*) \succeq S^i_{t-1}(s^*) \vee S^j_{t-1}(s^*)$. These two relations imply that $S' \vee S^j_{t-1}(s^*) = S^i_{t-1}(s^*) \vee S^j_{t-1}(s^*)$

We next show that $S^j_{t-1}(s^*_S) = S^j_{t-1}(s^*_{S'})$ and $P^j_{t-1}(s^*_S) = P^j_{t-1}(s^*_{S'})$. Since $S' \preceq S$ and from Projections Preserve Ignorance, $S^j_{t-1}(s^*_S) \succeq S^j_{t-1}(s^*_{S'})$. Also, $S^j_{t-1}(s^*) \succeq S^j_{t-1}(s^*_S)$ implies $S^j_{t-1}(s^*) \wedge S \succeq S^j_{t-1}(s^*_S) \wedge S = S^j_{t-1}(s^*_S)$ and $S' = (S^j_{t-1}(s^*) \wedge S) \vee S^i_{t-1}(s^*) \succeq S^j_{t-1}(s^*_S) \vee S^i_{t-1}(s^*) \succeq S^j_{t-1}(s^*_S)$. Again by Projections Preserve Ignorance, we have $S^j_{t-1}(s^*_{S'}) \succeq S^j_{t-1}(s^*_{S^j_{t-1}(s^*_S)}) = S^j_{t-1}(s^*_S)$. The last equality holds from Generalized Reflexivity and Stationarity. Finally, Stationarity and $S^j_{t-1}(s^*_S) = S^j_{t-1}(s^*_{S'})$ imply $P^j_{t-1}(s^*_S) = P^j_{t-1}(s^*_{S'})$.

The last equality implies that both s^*_S and $s^*_{S'}$ describe the same information and awareness about j. Since $s^*_S = s^*_{S^i_t(s^*)} \in F^y_{t-1}(S^i_t(s^*))$, it means that $s^*_{S^i_t(s^*)}$ can rationalise announcement y_{t-1}. But then this means that $s^*_{S'}$ can also rationalise y_{t-1}, so that $s^*_{S'} \in F^y_{t-1}(S')$. We know that $F^y_{t-1}(S^i_{t-1}(s^*)) \neq \emptyset$, as it rationalizes all announcement up to y_{t-2}. Because $S^i_{t-1}(s^*) \preceq S'$, we also have $F^y_{t-1}(S') \neq \emptyset$, therefore contradicting that $P^i_0(s^*)^{S'} \cap F^y_{t-1}(S') = \emptyset$ and that updating to $S^i_t(s^*) = S$ is minimal. $\square$

Proof of Theorem 1. Note that for all S and t, $F^y_{t+1}(S) \subseteq F^y_t(S)$. Although for some state space S and period t, we can have $F_t^y(S) = \emptyset$, by construction for the full state space $\overline{S}$ and any t we have that $F_t^y(\overline{S}) \neq \emptyset$. Since the collection of all states Σ is finite, there exists t such that $F_t^y(S) = F_{t'}^y(S)$ for all $S \in \mathcal{S}$ and $t' \geq t$.

Consider a state space S and time t such that $F_{t'}^y(S) \neq \emptyset$ for all $t' \geq t$ and $S' \prec S$ implies $F_t^y(S') = \emptyset$. Such S and t exist because of the finiteness of Σ, the fact that $F_t^y(\overline{S}) \neq \emptyset$ for all t and the fact that $\mathcal{S}$ is a lattice. Confinement and the fact that there is no less expressive state space than S imply that for all $s \in F_t^y(S)$ and $i \in I$, $P_t^i(s) \subseteq F_t^y(S) \subseteq S$. That is, $F_t^y(S)$ is partitioned by each P_t^i.

Note that each i announces the conditional expectation of X given her private information at s^* and the public information at each t'. Because the public information does not update after t, i repeats the same announcement y' for all $t' \geq t$. However, since each state $s \in F_{t'}^y(S)$, $t' \geq t$, is not excluded, it must be that i's announcement given each $s \in F_{t'}^y(S)$ is also y'. We therefore have that $E_{\pi'}[X|P_{t'}^i(s)] = y'$ for all $s \in F_{t'}^y(S)$. Integrating over $F_{t'}^y(S)$, we have that $E_{\pi'}[X] = y'$, where π' is the Bayesian update of the common generalized prior given the public information at each state space. Using the same arguments, it cannot be that another trader announces $y'' \neq y'$ at each $s \in F_{t'}^y(S)$, because this would imply $E_{\pi'}[X] = y''$, a contradiction. Hence, after t all traders agree on their announcement and $y_t = y_{t'}$ for all $t' \geq t$.

For the second claim, note that we have established in the proof of the first claim that for some t', for all $t \geq t'$, each trader i makes the same announcement y, where $E_{\pi'}[X|P_t^i(s_1^*)] = y$ for all $s_1^* \in F_t^y(\overline{S})$, and π' is the generalized prior which is the Bayesian update of π given $F_t^y(S)$ for each $S \in \mathcal{S}_t$.

Suppose that the true state is s_1^*. If for all $s^* \in \overline{S}$ with $\pi'(s^*) > 0$ we have $X(s^*) = v$, there is information aggregation at s_1^*. Suppose that for some $s^* \in \overline{S}$ with $\pi'(s^*) > 0$ we have $X(s^*) \neq v = y$. Since the security X is separable and setting $y = v$, condition (ii) of the definition of non-separability specifies that for some $s_1^* \in F_t^y(\overline{S})$, which is the support of π' on $\overline{S}$, we have $E_{\pi'}[X|P_t^i(s_1^*)] \neq v = y$. But this contradicts the result of the previous paragraph, that all states in $F_t^y(\overline{S})$ specify that all traders announce $y = v$.

□

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